Homework 10

This assignment is due by **11:59pm on Tuesday 11/26** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit.

1. Let $A = U\Sigma V^T$ be an SVD of a matrix A. Let $\mathbf{u}_i$ denote the ith column of U and let $\mathbf{v}_i$ denote the ith column of V. In class, we showed that $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$ for all $1 \leq i \leq r$ and $A\mathbf{v}_i = 0$ for all $r+1 \leq i \leq n$.
   (a) Show that $A^T\mathbf{u}_i = \sigma_i \mathbf{v}_i$ for all $1 \leq i \leq r$.
   (b) Show that $A^T\mathbf{u}_i = 0$ for $r+1 \leq i \leq m$.

2. Consider the following SVD of a matrix A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & \sqrt{3}/3 & -\sqrt{2}/6 & -\sqrt{6}/6 \\ -1/3 & 0 & 2\sqrt{2}/3 & 0 \\ 2/3 & \sqrt{3}/3 & \sqrt{2}/6 & -\sqrt{6}/6 \\ 0 & \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{bmatrix}^T$$

   (a) Determine the eigenvalues of A^TA. Explain your reasoning.
   (b) What is the rank of A? Explain your reasoning.
   (c) Determine orthonormal bases of the four fundamental subspaces of A. Explain your reasoning.
   (d) Diagonalize A^TA. Explain your reasoning.
   (e) Determine the SVD rank 1 approximation of A. Show your work.
   (f) What is the largest possible value of $||A\mathbf{u}||$, where $\mathbf{u}$ is a unit vector? Explain your reasoning.

3. Consider the following data points.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

(a) Describe how to find the 2-dimensional best fit subspace of the above data points (no computation required).

(b) Use Julia to find a basis for the 2-dimensional best fit subspace of the above data points. Include a screen shot of your code and output.

4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \\ 3 & 5 & 9 \end{bmatrix}$. Use Julia for the following calculations. Find an SVD for A and then compute the rank 1, 2, and 3 approximations for A and determine their Frobenius distances from A. Include a screenshot of your code and output.

   [the command "norm(M)" provides the Frobenius norm of a matrix M]

5. Suppose A is a 2×2 symmetric matrix with orthogonal diagonalization

$$A = Q \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} Q^T, \quad \text{where } Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}.$$

   Find an SVD of A.

6. Show that if the SVD of a square matrix A is also a diagonalization of A, then A is symmetric and positive semi-definite.

   [Note: the converse is also true!]